GAUSSIAN TYPE BOUNDS FOR THE NEUMANN-GREEN FUNCTION OF A GENERAL PARABOLIC OPERATOR

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Abstract. Based on the fact that the Neumann-Green function can be constructed as a perturbation of the fundamental solution by a single-layer potential, we establish a gaussian lower bound and a gaussian type upper bound for the Neumann-Green function for a general parabolic operator. We build our analysis on old tools coming from the construction of a fundamental solution of a general parabolic operator by means of the so-called parametrix method. At the same time we provide a simple proof for the gaussian two-sided bounds for the fundamental solution.

Key words : parabolic operator, fundamental solution, Neumann-Green function, parametrix, heat kernel.

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Contents

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with $C^{1,\alpha}$-boundary, where $0 < \alpha < 1$. Let $t_0 < t_1$, set $Q = \Omega \times (t_0, t_1)$ and consider the second order differential operator

$$L = a_{ij}(x,t)\partial^2_{ij} + b_k(x,t)\partial_k + c(x,t) - \partial_t.$$

Here and henceforth we use the usual Einstein's summation convention.

We make the following assumptions on the coefficients of L:

(i) the matrix $(a_{ij}(x,t))$ is symmetric for any $(x,t) \in \overline{Q}$,

(ii) $a_{ij} \in W^{1,\infty}(Q)$, $b_k,\ c \in C([t_0,t_1], C^\alpha(\overline{\Omega}))$,

(iii) $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $(x,t) \in \overline{Q}$, $\xi \in \mathbb{R}^n$,

(iv) $\|a_{ij}\|_{W^{1,\infty}(Q)} + \|b_k\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)} \leq A$,

where $\lambda > 0$ and $A > 0$ are two given constants.

Since we will use the fundamental solution in the whole space, we begin by extending the coefficients of L in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ to coefficients having the same regularity. We observe that this is possible in view of the regularity of Ω. For simplicity we keep the same symbols for the extended coefficients. We may also assume that the ellipticity condition holds for the extended coefficients with the same λ. Pick $\psi \in C_0^\infty(\widetilde{\Omega})$ satisfying $0 \leq \psi \leq 1$ and $\psi = 1$ in a neighborhood of $\overline{\Omega}$. We set

$$\widetilde{a}_{ij} = a_{ij}\psi + \lambda\delta_{ij}(1-\psi),\ \ \widetilde{b}_k = b_k\psi,\ \ \widetilde{c} = c\psi$$

and

$$\widetilde{L} = \widetilde{a}_{ij}(x,t)\partial_{ij}^2 + \widetilde{b}_k(x,t)\partial_k + \widetilde{c}(x,t) - \partial_t.$$

Clearly the coefficients of $\widetilde{L}$ satisfy the same assumptions as those of L. So in the sequel we will use the same symbol L for L or its extension $\widetilde{L}$.

We are interested in gaussian two-sided bounds for the Neumann-Green function associated to the operator L. More specifically, denoting by G the Neumann-Green function for L, we want to prove an estimate of the form

$$[C(t-\tau)]^{-n/2}e^{-C\frac{|x-\xi|^2}{t-\tau}} \leq G(x,t;\xi,\tau) \leq [\widetilde{C}(t-\tau)]^{-n/2}e^{-\widetilde{C}\frac{|x-\xi|^2}{t-\tau}},\ \ (x,t;\xi,\tau)\in Q^2\cap\{t>\tau\},$$

where the constants C and $\widetilde{C}$ depend only on Ω, t_0, t_1 and A.

We succeed in proving that the above gaussian lower bound holds true. But we are only able to prove an upper bound which is weaker than a gaussian upper bound. Namely we prove an upper bound of the form:

$$((t-\tau)^{1/2} + |x-\xi|)G(x,t,\xi,\tau) \leq [C(t-\tau)]^{-n/2}e^{-\frac{C|x-\xi|^2}{t-\tau}},\ \ (x,t;\xi,\tau)\in Q^2,\ t>\tau.$$

for some constant C that can depend only on Ω, t_0, t_1 and A.

It is an open problem to know whether the gaussian upper bound is true for a general smooth bounded domain Ω. We will see in Section 4 how the two-sided gaussian bounds, for the Neumann-Green function, can be obtained in a straightforward way from the gaussian two-sided bounds for the fundamental solution when Ω is a half space.

To our knowledge these kind of gaussian estimates have never been established before. The situation is completely different for the Dirichlet-Green function since this later vanishes on the boundary. One can prove in a straightforward manner, with the help of the maximum principle, that the Dirichlet-Green function is non negative and dominated pointwise by the fundamental solution and so it has a gaussian upper bound. Aronson [Ar2] (Theorem 8 in page 670)[1] get an interior gaussian lower bound for the Dirichlet-Green function. Later Cho [Ch], Cho, Kim and Park [CKP] extended this result to a global weighted gaussian bounds involving the distance to the boundary.

When L has time-independent coefficients a fundamental solution or a Green function is reduced to a heat kernel. We mention that there is a wide literature dealing with gaussian bounds for heat kernels. We quote the following classical books: [Da], [Gr], [Ou] [Sa], [St], but of course there are many other references on the subject.

As we said in the summary, the main ingredient in our analysis relies on the classical construction of the fundamental solution by the so-called parametrix method. We revisit this construction in the next section and we derive from it the gaussian two-sided bounds for the fundamental solution. Special attention is paid to the dependence of the gaussian upper bound on the lower order coefficients of L. This dependence will be in the heart of the proof of the gaussian type upper bound for the Neumann-Green function since we will conjugate L with e^{ψ}, for some function ψ appropriately chosen. This is a classical argument to derive a gaussian upper bound from a Nash upper bound. Note however that this argument is not enough to get a gaussian upper bound for the Neumann-Green function. In our case we need to conjugate again with a special function together with a maximum principle argument in order to get a gaussian type upper bound. This is done in Section 4. Before, we prove in Section 3 a gaussian lower bound for the Neumann-Green function. To do so, we construct the Neumann-Green function as a perturbation of the fundamental solution by a single-layer potential. The gaussian lower bound is derived from an estimate for the kernel of the single-layer potential which is the key point in the proof.

[1]Let us observe that Theorem 8 in page 670 of [Ar2] can be used to extend the results of Section 3 of [FS] to a general parabolic operator. In other words, one can obtain a proof of a continuity theorem by Nash [Na] and the Moser's Harnack inequality [Mo] for a general parabolic operator, which is based on the two-sided gaussian bounds for the fundamental solution.

2. The parametrix method revisited

This section is concerned with gaussian two-sided bounds for the fundamental solution. For a systematic study of the fundamental solution we refer to the classical monographs by A. Friedman [Fr] and O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'tzeva [LSU].

In the sequel $P = \mathbb{R}^n \times (t_0, t_1)$.

We recall that a fundamental solution of $Lu = 0$ in P is a function $E(x, t; \xi, \tau)$ which is $C^{2,1}$ in $P^2 \cap \{t > \tau\}$, satisfies

$$LE(\cdot\,, \cdot; \xi, \tau) = 0 \text{ in } \mathbb{R}^n \times \{\tau < t \le t_1\}, \text{ for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$$

and, for any $f \in C_0(\mathbb{R}^n)$ [2],

$$\lim_{t \searrow \tau} \int_{\mathbb{R}^n} E(x, t; \xi, \tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.$$

The construction of a fundamental solution by the so-called parametrix method was initiated by E. E. Levi [Le]. Let $a = (a^{ij})$ be the inverse matrix of (a_{ij}), $|a|$ the determinant of a and

$$Z(x, t; \xi, \tau) = [4\pi(t - \tau)]^{-n/2} \sqrt{|a|} e^{-\frac{a(\xi,\tau)(x-\xi)\cdot(x-\xi)}{4(t-\tau)}}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

This function is called the parametrix. It satisfies

$$L_0 Z(\cdot\,, \cdot\,, \xi, \tau) = 0 \text{ in } \mathbb{R}^n \times \{\tau < t \le t_1\} \text{ for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[,$$

where

$$L_0 = a_{ij}(\xi, \tau)\partial^2_{ij} - \partial_t. \tag{2.1}$$

In the parametrix method we seek E, a fundamental solution of $Lu = 0$ in P, of the form

$$E(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_\tau^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma)\Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma, \tag{2.2}$$

where Φ is to be determined in order to satisfy $LE(\cdot\,, \cdot; \xi, \tau) = 0$ for any $(\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$.

Following Formulas (4.4) and (4.5) in page 14 of [Fr], Φ is given by the series

$$\Phi = \sum_{\ell=1}^\infty \Phi_\ell,$$

where $\Phi_1(x, t; \xi, \tau) = LZ(x, t; \xi, \tau)$ and

$$\Phi_{\ell+1}(x, t; \xi, \tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma)\Phi_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma, \quad \ell \ge 1.$$

Here for simplicity we write $LZ(x, t; \xi, \tau)$ instead of $[LZ(\cdot, \cdot, \xi, \tau)](x, t)$.

Let d_i, $1 \le i \le n$, given by

$$d_i = d_i(x, t; \xi, \tau) = -\frac{a^{ij}(\xi, \tau)(x_j - \xi_j)}{2(t - \tau)}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

Then

$$\partial_i Z = d_i Z, \quad \partial^2_{ij} Z = \left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] Z.$$

Therefore, taking into account (2.1), we get

$$LZ = LZ - L_0 Z = \left\{ (a_{ij}(x, t) - a_{ij}(\xi, \tau)) \left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] + b_k d_k + c \right\} Z.$$

[2] One can take a larger class of functions. Namely a class of continuous functions satisfying a certain growth condition at infinity (e.g. formulas (6.1) and (6.2) in page 22 of [Fr]).

We write $LZ = \Psi Z$, where

$$\Psi = (a_{ij}(x,t) - a_{ij}(\xi,\tau))\left[-\frac{a^{ij}(\xi,\tau)}{2(t-\tau)} + d_jd_i\right] + b_kd_k + c.$$

Let

$$M = \max_{i,j}\|a_{ij}\|_{W^{1,\infty}(Q)},\quad N = \max(\max_k \|b_k\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, 1).$$

Since

$$|d_i| \le \frac{M|x-\xi|}{t-\tau},$$
$$|a_{ij}(x,t) - a_{ij}(\xi,\tau)| \le M(|x-\xi| + t - \tau),$$

we have

$$|\Psi(x,t;\xi,\tau)| \le N\frac{1}{\sqrt{t-\tau}}P\left(\frac{|x-\xi|}{\sqrt{t-\tau}}\right). \tag{2.3}$$

Here P is a polynomial function of degree less or equal to three whose coefficients depend only on M.

Unless otherwise stated, all the constants we use now do not depend on N.

In light of (2.3) we obtain

$$|LZ| \le CN(t-\tau)^{-(n+1)/2}P(\eta)e^{-(\lambda/4)|\eta|^2} = CN(t-\tau)^{-(n+1)/2}\left[P(\eta)e^{-(\lambda/8)|\eta|^2}\right]e^{-(\lambda/8)|\eta|^2},$$

with

$$\eta = \frac{|x-\xi|}{\sqrt{t-\tau}}.$$

But the function $\rho \in (0,+\infty) \longrightarrow P(\rho)e^{-(\lambda/8)\rho^2}$ is bounded. Consequently, where $\lambda^* = \lambda/8$,

$$|\Phi_1(x,t;\xi,\tau)| = |LZ(x,t;\xi,\tau)| \le N\widetilde{C}(t-\tau)^{-(n+1)/2}e^{-\frac{\lambda^*|x-\xi|^2}{t-\tau}}. \tag{2.4}$$

The following lemma will be useful in the sequel. Its proof is given in page 15 of [Fr].

Lemma 2.1. *Let $c > 0$ and $-\infty < \gamma, \beta < n/2 + 1$. Then*

$$\int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-\gamma}e^{-\frac{c|x-\eta|^2}{t-\sigma}}(\sigma-\tau)^{-\beta}e^{-\frac{c|\eta-\xi|^2}{\sigma-\tau}}d\eta d\sigma$$
$$= \left(\frac{4\pi}{c}\right)^{n/2} B(n/2-\gamma+1, n/2-\beta+1)(t-\tau)^{n/2+1-\gamma-\beta}e^{-\frac{c|x-\xi|^2}{t-\tau}},$$

where B is the usual beta function.

We want to show

$$|\Phi_\ell(x,t;\xi,\tau)| \le (N\widetilde{C})^\ell\widehat{C}^{\ell-1}(t-\tau)^{-(n+2-\ell)/2}\prod_{j=1}^{\ell-1}B(1/2,j/2)e^{-\frac{\lambda^*|x-\xi|^2}{t-\tau}},\quad \ell \ge 2. \tag{2.5}$$

Here $\widetilde{C}$ is the same constant as in (2.4) and $\widehat{C} = \left(\frac{4\pi}{\lambda^*}\right)^{n/2}$.

As

$$\Phi_2(x,t;\xi,\tau) = \int_\tau^t\int_{\mathbb{R}^n}\Phi_1(x,t;\eta,\sigma)\Phi_1(\eta,\sigma;\xi,\tau)d\eta d\sigma,$$

estimate (2.4) and Lemma 2.1 with $\gamma = \beta = n/2+1$ show that (2.5) holds true with $\ell = 2$. The general case follows by an induction argument in ℓ. Indeed, using

$$\Phi_{\ell+1}(x,t;\xi,\tau) = \int_\tau^t\int_{\mathbb{R}^n}\Phi_1(x,t;\eta,\sigma)\Phi_\ell(\eta,\sigma;\xi,\tau)d\eta d\sigma,$$

(2.4), (2.5) for ℓ and Lemma 2.1 with $\gamma = n/2+1$ and $\beta = (n+2-\ell)/2$, we obtain easily that (2.5) holds true with $\ell+1$ in place of ℓ.

If Γ is the usual gamma function, we recall that

$$B(1/2, j/2) = \frac{\Gamma(1/2)\Gamma(j/2)}{\Gamma((j+1)/2)}.$$

Therefore

$$\prod_{j=1}^{\ell-1} B(1/2, j/2) = \frac{\Gamma(1/2)^{\ell}}{\Gamma(\ell/2)} = \frac{\sqrt{\pi}^{\ell}}{\Gamma(\ell/2)}. \tag{2.6}$$

(2.4)-(2.6) entail that

$$|\Phi(x,t;\xi,\tau)| \leq \sum_{\ell \geq 1} |\Phi_{\ell}(x,t;\xi,\tau)| \leq \widetilde{C}(1+S)(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau}}, \tag{2.7}$$

with

$$S = \sum_{\ell \geq 1} \left[CN(t-\tau)^{1/2} \right]^{\ell} / \Gamma(\ell/2).$$

We have $\Gamma(\ell/2) = \Gamma(m) = (m-1)!$ if $\ell = 2m$ and $\Gamma(\ell/2) = \Gamma(m+1/2) \geq \Gamma(m) = (m-1)!$ if $\ell = 2m+1$. Then

$$\begin{aligned} S &= \sum_{m \geq 1} \left[CN(t-\tau)^{1/2} \right]^{2m} / \Gamma(m) + \sum_{m \geq 0} \left[CN(t-\tau)^{1/2} \right]^{2m+1} / \Gamma(2m+1/2) \\ &\leq \sum_{m \geq 1} \left[CN(t-\tau)^{1/2} \right]^{2m} / (m-1)! + CN(t-\tau)^{1/2} + \sum_{m \geq 1} \left[CN(t-\tau)^{1/2} \right]^{2m+1} / (m-1)!. \end{aligned}$$

Whence

$$S \leq \widetilde{C} e^{\widetilde{C} N^2 (t-\tau)}.$$

Plugging this estimate into (2.7), we obtain

$$|\Phi(x,t;\xi,\tau)| \leq \widetilde{C}(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \widetilde{C} N^2 (t-\tau)}. \tag{2.8}$$

With the help of Lemma 2.1, estimate (2.8) yields

$$\left| \int_{\tau}^{t} \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma)\Phi(\eta,\sigma;\xi,\tau) d\eta d\sigma \right| \leq \widetilde{C}(t-\tau)^{-(n-1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \widetilde{C} N^2 (t-\tau)}. \tag{2.9}$$

We obtain as a consequence of this inequality

$$|E(x,t;\xi,\tau)| \leq \widetilde{C}(t-\tau)^{-n/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \widetilde{C} N^2 (t-\tau)}. \tag{2.10}$$

This estimate is essential when establishing the gaussian type upper bound for the Neumann-Green function.

In the rest of this section we forsake the explicit dependence on N. So the constants below may depend on Ω, λ, A, t_0 and t_1.

From (2.9) we deduce in a straightforward manner

$$E(x,t;\xi,\tau) \geq \widehat{C}(t-\tau)^{-n/2}, \;\; (x,t;\xi,\tau) \in P^2, \; t > \tau, \; \widetilde{C}|x-\xi|^2 < t-\tau. \tag{2.11}$$

By Theorem 11 in page 44 of [Fr], E is positive. Moreover E satisfies the following identity, usually called the reproducing property,

$$E(x,t;\xi,\tau) = \int_{\mathbb{R}^n} E(x,t;\eta,\sigma)E(\eta,\sigma;\xi,\tau) d\eta, \;\; x, \xi \in \mathbb{R}^n, \;\; t_0 \leq \tau < \sigma < t \leq t_1. \tag{2.12}$$

We can now paraphrase the proof of Theorem 2.7 in page 334 of [FS] to get the gaussian lower bound for E.

We sum up our analysis in the following theorem.

Theorem 2.1. *The fundamental solution E satisfies the gaussian two-sided bounds:*

$$[C(t-\tau)]^{-n/2} e^{-C\frac{|x-\xi|^2}{t-\tau}} \leq E(x,t;\xi,\tau) \leq [\widetilde{C}(t-\tau)]^{-n/2} e^{-\widetilde{C}\frac{|x-\xi|^2}{t-\tau}}, \; (x,t,\xi,\tau) \in P^2 \times \{t > \tau\}. \tag{2.13}$$

Remark 2.1. Let us observe that the proof of Theorem 2.1 presented here is elementary in comparison with that given in [FS] for proving the gaussian two-sided bounds for the operator $\partial_i(a_{ij}(x,t)\partial_j\,\cdot\,)-\partial_t$ with (C^∞) smooth coefficients. Of course, Theorem 2.1 applies the operators in divergence form. Note however that the constants C and $\widetilde{C}$ appearing in (2.13) do not depend on t_0 and t_1 in the gaussian two-sided bounds established in [FS]. We mention that gaussian two-sided bounds were obtained by S. D. Eidel'man and F. O. Porper [EP] when the coefficients of L satisfy the uniform Dini condition with respect to x. The main tool in [EP] is a parabolic Harnack inequality. We refer also to [Ar1], [Fa], [It] and [NS], where the reader can find various results on bounds for the fundamental solution.

3. Gaussian lower bound for the Neumann-Green function

The unit outward normal vector at $x\in\partial\Omega$ is denoted by $\nu=\nu(x)$. Henceforth $\Sigma=\partial\Omega\times(t_0,t_1)$.

For $\tau\in[t_0,t_1[$, set $Q_\tau=\Omega\times(\tau,t_1)$ and $\Sigma_\tau=\partial\Omega\times(\tau,t_1)$. We consider the Neumann initial-boundary value problem (IBVP in short) for the operator L:

$$\left\{\begin{array}{ll} Lu=0 & \text{in } Q_\tau,\\ u(\cdot\,,\tau)=\psi & \text{in } \Omega,\\ \partial_\nu u=0 & \text{on } \Sigma_\tau. \end{array}\right. \tag{3.1}$$

From Theorem 2 in page 144 of [Fr] and its proof, for any $\psi\in C_0^\infty(\Omega)$, the IBVP (3.1) has a unique solution $u\in C(\overline{Q_\tau})\cap C^{2,1}(Q_\tau)$ given by

$$u(x,t)=\int_\tau^t\int_{\partial\Omega}E(x,t;\xi,\sigma)\varphi(\xi,\sigma)d\xi d\sigma+\int_\Omega E(x,t;\xi,\tau)\psi(\xi)d\xi. \tag{3.2}$$

Here

$$\varphi(x,t)=2F_\tau(x,t)+2\sum_{\ell\geq 1}\int_\tau^t\int_{\partial\Omega}M_\ell(x,t;\xi,\sigma)F_\tau(\xi,\sigma)d\xi d\sigma, \tag{3.3}$$

with

$$\begin{aligned} &F_\tau(x,t)=\int_\Omega\partial_\nu E(x,t;\xi,\tau)\psi(\xi)d\xi,\\ &M_1=2\partial_\nu E,\\ &M_{\ell+1}(x,t;\xi,\tau)=\int_\tau^t\int_{\partial\Omega}M_1(x,t;\eta,\sigma)M_\ell(\eta,\sigma;\xi,\tau)d\eta d\sigma. \end{aligned}$$

Let, where $(x,t)\in\Sigma_\tau$ and $\xi\in\Omega$,

$$N(x,t;\xi,\tau)=2\partial_\nu E(x,t;\xi,\tau)+2\sum_{\ell\geq 1}\int_\tau^t\int_{\partial\Omega}M_\ell(x,t;\eta,\sigma)\partial_\nu E(\eta,\sigma;\xi,\tau)d\eta d\sigma.$$

Assume for the moment (see the proof below) that

$$\varphi(x,t)=\int_\Omega N(x,t;\xi,\tau)\psi(\xi)d\xi. \tag{3.4}$$

We set

$$G(x,t,\xi,\tau)=\int_\tau^t\int_{\partial\Omega}E(x,t;\eta,\sigma)N(\eta,\sigma;\xi,\tau)d\eta d\sigma+E(x,t;\xi,\tau). \tag{3.5}$$

It follows from Fubini's theorem that

$$u(x,t)=\int_\Omega G(x,t;\xi,\tau)\psi(\xi)d\xi. \tag{3.6}$$

The function G is called the Neumann-Green function for $Lu=0$ in Q.

We have, for any $0 \leq \psi \in C_0^\infty(\Omega)$, $u \geq 0$ according to the maximum principle (e.g. Theorem 2.9 and remarks following it in page 15 of [Li]) ; whence $G \geq 0$. From the uniqueness of the solution of the IBVP (3.1) we have also

$$\int_\Omega G(x,t;\xi,\tau)\psi(\xi)d\xi = \int_\Omega G(x,t,\eta,\sigma)d\eta \int_\Omega G(\eta,\sigma,\xi,\tau)\psi(\xi)d\xi \text{ for any } \psi \in C_0^\infty(\Omega),\ \tau < \sigma < t.$$

Therefore

$$G(x,t;\xi,\tau) = \int_\Omega G(x,t,\eta,\sigma)G(\eta,\sigma,\xi,\tau)d\eta,\ \tau < \sigma < t. \tag{3.7}$$

That is G has the reproducing property.

Let us observe that, when $c = 0$, G satisfies in addition

$$\int_\Omega G(x,t;\xi,\tau)d\xi = 1.$$

We shall need the following key lemma for establishing the gaussian lower bound for G.

Lemma 3.1. *For $1 - \alpha/2 < \mu < 1$, we have*

$$|N(x,t;\xi,\tau)| \leq C(t-\tau)^{-\mu}|x-\xi|^{-n+1},\ (x,t) \in \Sigma_\tau,\ \xi \in \Omega,\ x \neq \xi. \tag{3.8}$$

The lemma below appears in page 137 of [Fr] as Lemma 1. It is needed for proving Lemma 3.1.

Lemma 3.2. *Let $0 \leq a, b < n-1$. Then*

$$\int_{\partial\Omega} |x-\eta|^{-a}|\eta-\xi|^{-b}d\eta \leq \widehat{C}|x-\xi|^{n-1-(a+b)}. \tag{3.9}$$

Proof of Lemma 3.1. Let $1 - \alpha/2 < \mu < 1$ be given. From formula (2.12) in page 137 of [Fr], we have

$$|\partial_\nu E(x,t;\xi,\tau)| \leq C(t-\tau)^{-\mu}|x-\xi|^{-n+1+(2\mu+\alpha-2)}$$

and then

$$|M_1(x,t;\xi,\tau)| \leq C(t-\tau)^{-\mu}|x-\xi|^{-n+1+(2\mu+\alpha-2)}. \tag{3.10}$$

Since

$$|M_2(x,t;\xi,\tau)| \leq \int_\tau^t \int_{\partial\Omega} |M_1(x,t;\eta,\sigma)||M_1(\eta,\sigma;\xi,\tau)|d\eta d\sigma,$$

(3.10) leads

$$|M_2(x,t;\xi,\tau)| \leq C^2 \int_\tau^t (t-\sigma)^{-\mu}(\tau-\sigma)^{-\mu}d\sigma \int_{\partial\Omega} |x-\eta|^{-n+1+(2\mu+\alpha-2)}|\xi-\eta|^{-n+1+(2\mu+\alpha-2)}d\eta. \tag{3.11}$$

Or from Lemma 3.2

$$\int_{\partial\Omega} |x-\eta|^{-n+1+(2\mu+\alpha-2)}|\xi-\eta|^{-n+1+(2\mu+\alpha-2)}d\eta \leq \widehat{C}|x-\xi|^{-n+1+2(2\mu+\alpha-2)}. \tag{3.12}$$

On the other hand

$$\int_\tau^t (t-\sigma)^{-\mu}(\tau-\sigma)^{-\mu}d\sigma = (t-\tau)^{-\mu+(1-\mu)} \int_0^1 s^{-\mu}(1-s)^{1-\mu}ds = (t-\tau)^{-\mu+(1-\mu)}B(1-\mu,1-\mu). \tag{3.13}$$

We plug (3.12) and (3.13) into (3.11) ; we obtain

$$|M_2(x,t;\xi,\tau)| \leq C^2\widehat{C}(t-\tau)^{-\mu+(1-\mu)}B(1-\mu,1-\mu)|x-\xi|^{-n+1+2(2\mu+\alpha-2)}.$$

Now an induction argument in ℓ yields

$$|M_\ell(x,t;\xi,\tau)| \leq C^\ell\widehat{C}^{\ell-1}(t-\tau)^{-\mu+(\ell-1)(1-\mu)}\frac{\Gamma(1-\mu)^\ell}{\Gamma(\ell(1-\mu))}|x-\xi|^{-n+1+\ell(2\mu+\alpha-2)}.$$

It follows from $\Gamma(\ell(1-\mu)) \geq [\ell(1-\mu)]!$ (here $[\cdot]$ is the entire part) that the series

$$S(x,t;\xi,\tau) = \sum_{\ell\geq 1} C^\ell \widehat{C}^{\ell-1}(t-\tau)^{(\ell-1)(1-\mu)} \frac{\Gamma(1-\mu)^\ell}{\Gamma(\ell(1-\mu))}|x-\xi|^{\ell(2\mu+\alpha-2)}$$

converges uniformly in all its arguments. We complete the proof by noting that

$$N(x,t;\xi,\tau) = (t-\tau)^{-\mu}|x-\xi|^{-n+1}S(x,t;\xi,\tau).$$

□

Proof of (3.4). Let

$$N_k(x,t;\xi,\tau) = 2\partial_\nu E(x,t;\xi,\tau) + 2\sum_{\ell\geq 1}^{k}\int_\tau^t\int_{\partial\Omega} M_\ell(x,t;\eta,\sigma)\partial_\nu E(\eta,\sigma;\xi,\tau)d\eta d\sigma.$$

$$\varphi_k(x,t) = 2F_\tau(x,t) + 2\sum_{\ell\geq 1}^{k}\int_\tau^t\int_{\partial\Omega} M_\ell(x,t;\xi,\sigma)F_\tau(\xi,\sigma)d\xi d\sigma.$$

In light of Lemma 3.2 and with the help Lebesgue's dominated convergence theorem we can assert that

$$\int_\Omega N_k(x,t;\xi,\tau)\psi(\xi)d\xi \longrightarrow \int_\Omega N(x,t;\xi,\tau)\psi(\xi)d\xi \text{ as } k\longrightarrow +\infty.$$

Or according to Funini's theorem

$$\varphi_k(x,t) = \int_\Omega N_k(x,t;\xi,\tau)\psi(\xi)d\xi.$$

But $\varphi_k(x,t)\to\varphi(x,t)$ when k tends to infinity. Then the uniqueness of the limit yields

$$\varphi(x,t) = \int_\Omega N(x,t;\xi,\tau)\psi(\xi)d\xi.$$

□

We are now ready to prove

Theorem 3.1. *The Green function G satisfies the gaussian lower bound:*

$$[C(t-\tau)]^{-n/2}e^{-C\frac{|x-\xi|^2}{t-\tau}} \leq G(x,t;\xi,\tau),\;\; (x,t;\xi,\tau)\in Q^2\cap\{t>\tau\}. \tag{3.14}$$

Proof. Let

$$G_0(x,t;\xi,\tau) = \int_\tau^t\int_{\partial\Omega} E(x,t;\eta,\sigma)N(\eta,\sigma;\xi,\tau)d\xi d\sigma.$$

Let $0<\beta<1$ be fixed. From the gaussian upper bound for E we obtain in a straightforward way that

$$|E(x,t;\xi,\tau)| \leq C(t-\tau)^{-\beta}|x-\xi|^{-n+2\beta}.$$

This estimate together with (3.8) imply

$$\begin{aligned}|G_0(x,t;\xi,\tau)| &\leq \widetilde{C}\int_\tau^t (t-\sigma)^{-\mu}(\sigma-\tau)^{-\beta}d\sigma\int_{\partial\Omega}|x-\eta|^{-n+1}|\eta-\xi|^{-n+2\beta}d\eta\\ &\leq \widetilde{C}(t-\tau)^{1-\mu-\beta}|x-\xi|^{-n+2\beta},\end{aligned}$$

where we used Lemma 3.2. That is

$$|G_0(x,t;\xi,\tau)| \leq \widetilde{C}(t-\tau)^{-n/2+1-\mu}[(t-\tau)^{-1}|x-\xi|^2]^{-n/2+\beta}. \tag{3.15}$$

But we already know from (2.11) that

$$E(x,t;\xi,\tau) \geq C(t-\tau)^{-n/2},\;\; (x,t;\xi,\tau)\in P^2,\; t>\tau,\; \widehat{C}|x-\xi|^2 < t-\tau.$$

Hence

$$\begin{aligned} G(x,t;\xi,\tau) &\geq E(x,t;\xi,\tau) - |G_0(x,t;\xi,\tau)| \\ &\geq C(t-\tau)^{-n/2}(1-\widetilde{C}(t-\tau)^{1-\mu}),\ t>\tau,\ \widehat{C}|x-\xi|^2 < t-\tau. \end{aligned}$$

Consequently,

$$G(x,t;\xi,\tau) \geq C(t-\tau)^{-n/2},\ \ (x,t;\xi,\tau)\in Q^2,\ t>\tau,\ \widetilde{C}|x-\xi|^2 < t-\tau.$$

Or equivalently

$$G(x,t;\xi,\tau) \geq C(t-\tau)^{-n/2},\ \ (x,t;\xi,\tau)\in Q^2,\ t>\tau,\ |x-\xi| < \widehat{C}(t-\tau)^{1/2}. \tag{3.16}$$

Since $G \geq 0$ and satisfies (3.7), we proceed as in the proof of Theorem 2.1 to get the gaussian lower bound for G. Here we rewrite the arguments for the reader convenience. As Ω is connected, we find a path $\gamma : [0,1] \to \Omega$ connecting x to ξ which is piecewise constant. Let k be a positive integer and set $y_i = \gamma(i/k)$, $0 \leq i \leq k$. Then it is not difficult to show that there exists a constant $c \geq 1$ not depending on k such that

$$|y_{i+1} - y_i| \leq \frac{c}{k}|x-\xi|,\ \ 0 \leq i \leq k-1. \tag{3.17}$$

Shortening if necessary the constant $\widehat{C}$ in (3.16), we make the assumption

$$\widehat{C}(t-\tau)^{1/2} < 4\mathrm{dist}(\gamma([0,1]),\partial\Omega). \tag{3.18}$$

When $2c|x-\xi| \leq \widehat{C}(t-\tau)^{1/2}$ (implying $|x-\xi| \leq \widehat{C}(t-\tau)^{1/2}$), (3.14) follows immediately from (3.16). Therefore we may assume that $2c|x-\xi| > \widehat{C}(t-\tau)^{1/2}$. We choose $m \geq 2$ as the smallest integer satisfying

$$2c\frac{|x-y|}{m^{1/2}} \leq \widehat{C}(t-\tau)^{1/2}.$$

Set $x_i = \gamma(i/m)$, $0 \leq i \leq m$, and

$$r = \frac{1}{4}\widehat{C}\left(\frac{t-\tau}{m}\right)^{1/2}.$$

In light of the reproducing property and the positivity of G we obtain[3]

$$\begin{aligned} G(x,t;\xi,\tau) &= \int_\Omega \dots \int_\Omega G\left(x,t;\xi_1,\frac{(m-1)t+\tau}{m}\right)\dots G\left(\frac{t+(m-1)\tau}{m},\xi_{m-1};\xi,\tau\right)d\xi_1\dots d\xi_{m-1} \\ &\geq \int_{B(x_1,r)}\dots\int_{B(x_{m-1},r)} G\left(x,t;\xi_1,\frac{(m-1)t+\tau}{m}\right)\dots G\left(\frac{t+(m-1)\tau}{m},\xi_{m-1};\xi,\tau\right)d\xi_1\dots d\xi_{m-1}. \end{aligned} \tag{3.19}$$

Let $\xi_0 = x$ and $\xi_m = \xi$; we have

$$|\xi_{i+1}-\xi_i| \leq |x_{i+1}-x_i| + 2r \leq c\frac{|x-\xi|}{m} + 2r \leq c\frac{|x-\xi|}{m^{1/2}} + 2r \leq 4r,\ \ 0 \leq i \leq m-1.$$

Whence

$$|\xi_{i+1}-\xi_i| \leq \widehat{C}\left(\frac{t-\tau}{m}\right)^{1/2},\ \ 0 \leq i \leq m-1.$$

[3]Note that $B(x_i,r) \subset \Omega$, $1 \leq i \leq m-1$, as a consequence of (3.18)

It follows from (3.16) that

$$\begin{aligned}G(x,t;\xi,\tau) &\geq \int_{B(x_1,r)}\dots\int_{B(x_{m-1},r)} C^m\left(\frac{t-\tau}{m}\right)^{-nm/2} d\xi_1\dots d\xi_{m-1}\\ &\geq \omega_n^{m-1}r^{n(m-1)}C^m\left(\frac{t-\tau}{m}\right)^{-nm/2}\\ &\geq \omega_n^{m-1}\left[\frac{\widehat{C}^2}{16}\left(\frac{t-\tau}{m}\right)\right]^{n(m-1)/2} C^m\left(\frac{t-\tau}{m}\right)^{-nm/2}\\ &\geq \widetilde{C}C^m(t-\tau)^{-1/2},\end{aligned}$$

where ω_n is the measure of the unit ball of $\mathbb{R}^n$. In particular

$$G(x,t;\xi,\tau) \geq \widetilde{C}e^{-Cm}(t-\tau)^{-1/2}. \tag{3.20}$$

Or from the definition of m we have

$$m-1 \leq \left(\frac{2c}{\widehat{C}}\right)^2\frac{|x-y|^2}{t-\tau}. \tag{3.21}$$

Finally, a combination of (3.20) and (3.21) leads to (3.14). □

We mentioned previously that, when L has time-independent coefficients, the Neumann-Green function in nothing else but the heat kernel associated to the semi-group generated by L under the Neumann boundary condition.

To this end we assume that Ω is of class C^2 [4], $t_0 = 0$, $t_1 = T$ and the coefficients of the operator L are time-independent. In other words L is of the form

$$L = a_{ij}(x)\partial_{ij}^2 + b_k(x)\partial_k + c(x) - \partial_t \tag{3.22}$$

and the following assumption holds true:

(i') the matrix $(a_{ij}(x))$ is symmetric for any $x \in \overline{\Omega}$,

(ii') $a_{ij} \in W^{1,\infty}(\Omega)$, $b_k,\ c \in C^\alpha(\overline{\Omega})$,

(iii') $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $(x,t) \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$,

(iv') $\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \|b_k\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq A$,

where $\lambda > 0$ and $A > 0$ are two given constants.

Under the above mentioned assumptions, it is known (e.g. [Ou]) that the operator

$$A_L = a_{ij}(x)\partial_{ij}^2 + b_k(x)\partial_k + c(x),\ \text{ with domain } D(A_L) = \{u \in H^2(\Omega);\ \partial_\nu u = 0 \text{ on } \partial\Omega\},$$ [5]

generates an analytic semi-group e^{tA_L} on $L^2(\Omega)$.

Let $\psi \in C_0^\infty(\Omega)$. Since $u(t) = e^{tA_L}\psi$ is the solution of the IBVP (3.1),

$$e^{tA_L}\psi = \int_\Omega G(x,t;\xi,0)\psi(\xi)d\xi,\ 0 < t \leq T.$$

We rewrite this equality as follows:

$$e^{tA_L}\psi = \int_\Omega K(x,\xi,t)\psi(\xi)d\xi,\ 0 < t \leq T.$$

[4] This assumption is not really necessary, we make it just because in this case the domain of the operator A_L below is a subset of $H^2(\Omega)$ and therefore the normal derivative of an element of $D(A_L)$ exists in the usual trace sense.

[5] Notice that A_L can be rewritten as an operator in divergence form simply by observing that

$$a_{ij}(x)\partial_{ij}^2 = \partial_j(a_{ij}(x)\partial_j\cdot) - \partial_j a_{ij}(x)\partial_j.$$

The function

$$K(x,\xi,t)=G(x,t;\xi,0).$$

is usually called the heat kernel of the semi-group e^{tA_L}.

A straightforward consequence of Theorems 3.1 is

Corollary 3.1. *The Neumann heat kernel K satisfies the gaussian lower bound:*

$$(Ct)^{-n/2}e^{-\frac{C|x-\xi|^2}{t}}\leq K(x,\xi,t),\ \ (x,\xi)\in\Omega^2,\ 0<t\leq T. \tag{3.23}$$

The gaussian lower bound for the Neumann heat kernel was proved in [COY] when L is the laplacien. A quick examination of the proof show that this result can be extended to an operator of the form (3.22) with C^∞-smooth coefficients. The key point is the Hölder continuity of $x\longrightarrow K(x,\xi,t)$ which relies on the fact that $\mu-A_L$ is an isomorphism from $H^s(\Omega)$ into $H^{s-2}(\Omega)$, for some large μ and s of order $n/2+1$. This explain why the approach in [COY] can not be used to extend the lower gaussian bound for operators with less smooth coefficients.

4. Gaussian type upper bound for the Neumann-Green function

In this section, Ω is assumed to be of class $C^{2,\alpha}$.

We start with the so-called Nash upper bound.

Lemma 4.1. *The Green function G satisfies the Nash upper bound:*

$$G(x,t;\xi,\tau)\leq C(t-\tau)^{-n/2},\ \ (x,t;\xi,\tau)\in Q^2,\ t>\tau. \tag{4.1}$$

Proof. From the preceding proof we have, for any $0<\beta<1$,

$$G(x,t;\xi,\tau)\leq C(t-\tau)^{-\beta}|x-\xi|^{-n+2\beta}. \tag{4.2}$$

Or by (3.7) (reproducing property)

$$G(x,t;y,\tau)=\int_\Omega G(x,t,\xi,(t-\tau)/2)G(\xi,(t-\tau)/2,y,\tau)d\xi. \tag{4.3}$$

In light of the inequality in Lemma 2 in page 14 of [Fr][6], it follows from (4.2) and (4.3)

$$G(x,t;\xi,\tau)\leq C(t-\tau)^{-2\beta}|x-\xi|^{-n+2(2\beta)}.$$

We find by repeating this argument k-times

$$G(x,t;\xi,\tau)\leq C(t-\tau)^{-k\beta}|x-\xi|^{-n+k(2\beta)}. \tag{4.4}$$

Let k be the smallest integer satisfying $n<2k$. Then the choice $\beta=n/(2k)$ in (4.4) gives (4.1). □

Let $\zeta\in\mathbb{R}^n$ and set $|\nu|_\infty=\||\nu|\|_\infty$. As Ω is of class $C^{2,\alpha}$, we obtain from Lemma 3.1 in [Cho] that there exists $\theta_\zeta\in C^{2,\alpha}(\overline{\Omega})$ having the properties

$$\theta_\zeta\geq 1,\quad \theta_\zeta=1 \text{ on } \partial\Omega,\quad -\partial_\nu\theta_\zeta\geq|\nu|_\infty|\zeta| \text{ on } \partial\Omega. \tag{4.5}$$

Moreover examining the proof of Lemma 3.1 in [Cho] we obtain in a straightforward manner that

$$|\partial_i\theta_\zeta|,\ |\partial_{ij}\theta_\zeta|\leq\varrho|\zeta|, \tag{4.6}$$

where ϱ is a constant independent on ζ.

We now conjugate L by $e^{\zeta\cdot x+K|\zeta|^2t}\theta_\zeta$, where K is a constant to be specified later. To this end we consider

$$L^\zeta=e^{-\zeta\cdot x-K|\zeta|^2t}\theta_\zeta^{-1}Le^{\zeta\cdot x+K|\zeta|^2t}\theta_\zeta.$$

[6] Let $0<\alpha,\beta<n$. Then there exists a positive constant C such that for any $(x,\xi)\in\Omega^2$, $x\neq\xi$, we have

$$\int_\Omega|x-\eta|^{-\alpha}|\eta-\xi|^{-\beta}d\eta\leq C|x-\xi|^{n-\alpha-\beta} \text{ if } \alpha+\beta>n \ \text{ and } \ \int_\Omega|x-\eta|^{-\alpha}|\eta-\xi|^{-\beta}d\eta\leq C \text{ if } \alpha+\beta\leq n.$$

An elementary calculation gives $L^\zeta = \widehat{L}^\zeta + c^\zeta$ with

$$\widehat{L}^\zeta = a_{ij}\partial^2_{ij} + b_p\partial_p + 2a_{kl}(\zeta_l + \theta_\zeta^{-1}\partial_l\theta_\zeta)\partial_k + b_m(\zeta_m + \theta_\zeta^{-1}\partial_m\theta_\zeta) + c + \theta_\zeta^{-1}a_{ij}\partial^2_{ij}\theta_\zeta - \partial_t$$

and

$$c^\zeta = a_{pq}\zeta_p\zeta_q + 2\theta_\zeta^{-1}a_{ij}\zeta_j\partial_i\theta_\zeta - K|\zeta|^2.$$

In light of (4.5) and (4.6) we can choose K, independent on ζ, such that $c^\zeta \le 0$ for any $\zeta \in \mathbb{R}^n$.

Let $0 \le \psi \in C_0^\infty(\Omega)$ and u the corresponding solution of the IBVP (3.1). Then $v = e^{-\zeta\cdot x - K|\zeta|^2 t}\theta_\zeta^{-1}u$ solves the following IBVP

$$\left\{\begin{array}{ll} L^\zeta v = 0 & \text{in } Q_\tau, \\ v(\cdot,\tau) = e^{-\zeta\cdot x - K|\zeta|^2\tau}\theta_\zeta^{-1}\psi & \text{in } \Omega, \\ \partial_\nu v + (\zeta\cdot\nu + \partial_\nu\theta_\zeta)v = 0 & \text{on } \Sigma_\tau. \end{array}\right. \tag{4.7}$$

Next, let $\widehat{v}$ be the solution of the IBVP

$$\left\{\begin{array}{ll} \widehat{L}^\zeta \widehat{v} = 0 & \text{in } Q_\tau, \\ \widehat{v}(\cdot,\tau) = e^{-\zeta\cdot x - K|\zeta|^2\tau}\theta_\zeta^{-1}\psi & \text{in } \Omega, \\ \partial_\nu \widehat{v} = 0 & \text{on } \Sigma_\tau. \end{array}\right. \tag{4.8}$$

Since $v \ge 0$,

$$\left\{\begin{array}{ll} \widehat{L}^\zeta(v-\widehat{v}) = -c^\zeta v \ge 0 & \text{in } Q_\tau, \\ (v-\widehat{v})(\cdot,\tau) = 0 & \text{in } \Omega, \\ \partial_\nu(v-\widehat{v}) = -(\zeta\cdot\nu + \partial_\nu\theta_\zeta)v \ge 0 & \text{on } \Sigma_\tau. \end{array}\right.$$

The last inequality is obtained from the third inequality in (4.5). We invoke Theorem 2.9 [Li] to deduce that $v \le \widehat{v}$.

Let $\widehat{E}^\zeta$ and $\widehat{G}^\zeta$ be respectively the fundamental solution and the Neumann-Green function corresponding to $\widehat{L}^\zeta$. It follows from estimate (2.10) that

$$\widehat{E}^\zeta(x,t;\xi,\tau) \le \widetilde{C}(t-\tau)^{-n/2}e^{-\frac{\lambda^*|x-\xi|^2}{t-\tau} + \widehat{C}(1+|\zeta|+|\zeta|^2)(t-\tau)}.$$

A slight modification of the proofs of Lemmas 3.1 and 4.1 leads to the following Nash upper bound for $\widehat{G}^\zeta$:

$$\widehat{G}^\zeta(x,t;\xi,\tau) \le C(t-\tau)^{-n/2}e^{\widehat{C}(1+|\zeta|+|\zeta|^2)(t-\tau)}. \tag{4.9}$$

Therefore, for any $0 \le \psi \in C_0^\infty(\Omega)$,

$$e^{-\zeta\cdot x - K|\zeta|^2 t}\theta_\zeta^{-1}(x)u(x,t) = v(x,t) \le \int_\Omega \widehat{G}^\zeta(x,t;\xi,\tau)e^{-\zeta\cdot\xi - K|\zeta|^2\tau}\theta_\zeta^{-1}(\xi)\psi(\xi)d\xi.$$

Or

$$u(x,t) = \int_\Omega G(x,t;\xi,\tau)\psi(\xi)d\xi.$$

Whence

$$G(x,t,\xi,\tau) \le e^{\zeta\cdot x + K|\zeta|^2 t}\widehat{G}^\zeta(x,t;\xi,\tau)e^{-\zeta\cdot\xi - K|\zeta|^2\tau}\theta_\zeta(x)\theta_\zeta^{-1}(\xi).$$

This and (4.9) imply

$$G(x,t,\xi,\tau) \le C(t-\tau)^{-n/2}e^{\zeta\cdot(x-\xi) + \widehat{C}(1+|\zeta|+|\zeta|^2)(t-\tau)}\theta_\zeta(x)\theta_\zeta^{-1}(\xi) \tag{4.10}$$

and then

$$G(x,t,\xi,\tau) \le C|\zeta|(t-\tau)^{-n/2}e^{\zeta\cdot(x-\xi) + \widehat{C}(1+|\zeta|+|\zeta|^2)(t-\tau)}.$$

Taking $\zeta = (\xi - x)/[2\widehat{C}(t-\tau)]$ in the last estimate we get the following result.

Theorem 4.1. *The Neumann-Green function satisfies the upper bound:*

$$((t-\tau)^{1/2} + |x-\xi|)G(x,t,\xi,\tau) \le [C(t-\tau)]^{-n/2}e^{-\frac{C|x-\xi|^2}{t-\tau}},\ \ (x,t;\xi,\tau) \in Q^2,\ t > \tau.$$

Remark 4.1. 1) Returning to the proof of Lemma 3.1 in [Cho], we prove that for any compact subset K of Ω there exists C_K such that

$$|\theta_\zeta(x)\theta_\zeta^{-1}(\xi)| \le C_K, \ \ x \in \overline{\Omega}, \ \xi \in K \text{ and } \zeta \in \mathbb{R}^n.$$

We derive from (4.10) the following gaussian upper bound

$$G(x,t,\xi,\tau) \le C_K(t-\tau)^{-n/2}e^{-\frac{C|x-\xi|^2}{t-\tau}}, \ \ (x,t;\xi,\tau) \in Q^2 \cap \{\xi \in K, \ t > \tau\}.$$

2) The construction of the Neumann-Green function is quite simple when Ω is a half space, say

$$\Omega = \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; \ x_n > 0\}.$$

Assume for simplicity that the original coefficients of L satisfy assumptions (i)-(iv) in the whole space $\mathbb{R}^n$ and set

$$G(x,t;\xi,\tau) = \frac{1}{2}\left[E(x,t;\xi,\tau) + E(x^s,t;\xi,\tau)\right], \ \ x, \, \xi \in \mathbb{R}^n_+, \ \tau, t \in (t_0,t_1), \ \tau < t,$$

where $x^s = (x', -x_n)$ if $x = (x', x_n)$.

It is not hard to check that G is the Neumann-Green function for L in $\mathbb{R}^n_+ \times (t_0,t_1)$. In addition observing that $|x-\xi| \le |x^s - \xi|$ for any $x, \xi \in \mathbb{R}^n_+$, we see the gaussian two-sided bounds for E yield the gaussian two-sided bounds for G.

For $1 < p, q \le \infty$, let $\| \cdot \|_{p,q}$ denotes the norm in $L^q(t_0,t_1;L^p(\Omega))$. We have similarly to Corollary 7.1 in page 668 of [Ar2] the following result which is a direct consequence of the upper bound in Theorem 4.1.

Corollary 4.1. *Let $1 < p, q \le \infty$ be such that*

$$\frac{n}{p} + \frac{2}{q} < 1.$$

Then

$$\|G(x,t;\cdot,\cdot\,)\|_{p',q'}, \ \ \|G(\cdot\,,\cdot;\xi,\tau)\|_{p',q'} \le C,$$

where p', q' are the respective Hölder conjugate exponents of p and q.

Let $f \in C_0^\infty(Q_\tau)$ and p, q be as in the previous lemma. Then the solution of the following BVP

$$\begin{cases} Lu = f & \text{in } Q_\tau, \\ u(\cdot\,,\tau) = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma_\tau \end{cases}$$

can be represented by the formula

$$u(x,t) = \int_\tau^t \int_\Omega G(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau.$$

In light of this formula we have the following immediate consequence of Corollary 4.1:

$$\|u\|_{\infty,\infty} \le C\|f\|_{p,q}.$$

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